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# ELEMENTARY NOTES.

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Under the above title I propose to give, without much regard to their order, short notes upon some of the fundamental processes and concepts that come into actual work or into teaching, including certain questions as to their best forms and expression or mode of presentation, their natural extension, etc. In such of the notes as are developments of ideas possibly first suggested either in Oliver, Wait, and Jones's Algebra, or at the Cornell University Mathematical Seminary, the references O. W. J. and C. M. S. may be used.

## I.

### GENERAL AND LOGICO-MATHEMATICAL NOTATION.

I have found all of the following principles and general devices to be very helpful in actual work, and some of them, also, in teaching.

1. *Accents and suffixes.*—It is often well to distinguish between two uses of these, viz:

One use is to mark such values or functions as are co-ordinate with those denoted by the unmodified symbols, or such as are derived therefrom without loss of generality; while the other use is to mark particular values or cases, or to impart additional information. The distinction sometimes cannot be observed, but often it promotes neatness and perspicuity, enabling each symbol to tell naturally its own tale.

Now, two considerations point to the assignment of accents (upper or lower, acute or grave) or of points, etc., to denote derived or co-ordinate values, and of subscript numbers for particular values. Thus, for instance, if $x, y$ be variables, and if $x', y'$ be $ax + by$, $mx + ny$, then the particular values $x_1$, $y_1$, $x'_1, \ldots$ (which may be regarded as constants) are connected by the relations $x'_1 = ax_1 + by_1, \ldots$.

($a$) Usage already shows a tendency in this direction. Thus exponents, root-indices, and indices of differentiation are written above their stems and simply modify without specializing the meaning of the stem-symbols; and again, it is common to express differentiation by points or accents, as $x$ for $\frac{dx}{dt}$, or $f''(x)$ for $\frac{\partial^2}{\partial x^2} f(x)$; and so, we may write $f^{\prime}_{\prime\prime}(x, y)$ for $\frac{\partial^3}{\partial x \partial y^2} f(x, y)$, $f^{\prime}_{\prime\prime}(y, x)$ for $\frac{\partial^3}{\partial y \partial x^2} f(y, x)$, etc.

($\beta$) Apparently, the proposed assignment oftenest avoids the undesirable combination of exponents with upper accents on the same stem.

The distinction here made may often help us in devising appropriate symbols, whether for permanent use or only for the moment. Thus, to exhibit when desired the positive or negative character of an expression, we should use suffixes (perhaps $_+$ or $_-$) because they are to give additional information without modifying the previous meaning and laws of the expression; while to denote that the expression had been *made* positive or negative, by reversing its sign if necessary (O. W. J., p. 5), a superior index (say $^+$ or $^-$) would be used.

In the series $x$, $x'$, $x''$, ..., or $x$, $x^{\backprime}$, ..., or $x$, $x_{\prime}$, ..., or etc., the initial term $x$ commonly has a certain unique character as compared with the other terms; and so, I think, should $x_0$ have, as compared with $x_1$, $x_2$, ..., or else $x_0$ should not be used; and so with zero values of the integers $r$, $s$ in the series $x_{rs}$, $x_{r|s}$, etc.

2. *Indices to copulas.*—(*a*) It is often convenient to modify the copula $=$ so as either (1) to make some other or less definite assertion than that of strict equality, or else (2) to give additional information. These two general cases are analogous to the correspondingly numbered cases in Art. 1. For instance, in the symbol $x'_3$, the accent only varies the information given, perhaps by referring to different axes from those of $x$ and $y$; but the suffix 3 makes that information more definite, though perhaps less general, by showing that the quantity indicated not only is an $x'$, but also is that particular value of $x'$, or is the $x'$ of the particular point $P_3$.

Let us therefore extend to copulas, (as $=$, $>$, etc.), the principle of Art. 1; i. e. let us modify the copula at need by indices, points, etc., in the general case (1), and by subscript numbers or letters in the general case (2).

For instance, using $=$ in the sense "Approaches in value to, as a limit," $U =_{\Delta x} C$ would so naturally mean "$U$ approaches $C$ as a limit, when $\Delta x$ approaches its limit (viz. 0)," that these words of explanation would hardly be needed.

So $=_3$ might mean "Is equal to, by reason of equation (or article) 3, above;" and the context would sufficiently show if this were the meaning.

As another illustration of how nearly these symbols may tell their own story when one has in mind the general subject-matter involved, we may write, as follows, a familiar cross-ratio proof of Pascal's hexagon-theorem:

Let $c$ be any conic; 123456 the vertices of any inscribed hexagon; and 7, 8, 9 the respective intersections 23.56, 61.34, 45.12; then are 7, 8, 9 collinear.

For $$(8\text{–}2673) =_{23} (6\text{–}2153) =_{c} (4\text{–}2153) =_{12} (8\text{–}2194) =_{8} (8\text{–}2693);$$

$$\therefore\ 7,\ 8,\ 9 \text{ are collinear.} \quad \text{Q. E. D.}$$

Here, as is customary, (8–2673) denotes the cross-ratio of the pencil 82, 86, 87, 83; and it almost goes without saying that $=_{23}$, $=_c$, and $=_8$ mean, respectively, "Is equal to, by reason of the transversal 23," and ". . . . of the conic $c$," and ". . . . of the pencil whose vertex is 8."

($b$) The copula $=$ admits of further specialization. Between its meaning as first used by me, "Is nearly equal to," and Byerly's modification "Approaches as a limit," and perhaps yet other useful shades of interpretation, the context will sufficiently determine; but another distinction affects its formal laws and therefore should be indicated. The statement $U = V$ may mean ($\alpha$) that $U - V$ approaches zero, or contains only terms beyond a certain order, or may be for the moment neglected; or it may mean ($\beta$) that $(U-V)/\sqrt{UV}$, $=\sqrt{U/V} - \sqrt{V/U}$, does so. Understood in the sense $\left\{\begin{matrix}\alpha\\ \beta\end{matrix}\right\}$, the statement $U \doteq V$ means that the $\left\{\begin{matrix}\text{absolute}\\ \text{relative}\end{matrix}\right\}$ error of taking $V$ for $U$ or $U$ for $V$ may be neglected; and hence the statement is equivalent to $\left\{\begin{matrix}U-V \doteq 0 \doteq V-U\\ U/V \doteq 1 \doteq V/U\end{matrix}\right\}$. The two senses are identical, except that when $U$ and $V$ approach $\left\{\begin{matrix}0\\ \infty\end{matrix}\right\}$ we may have $U = V$ in the sense $\left\{\begin{matrix}\alpha \text{ and not } \beta\\ \beta \text{ and not } \alpha\end{matrix}\right\}$. But, since the ratios of magnitudes are important oftener than the absolute values, let us keep $\doteq$ in the sense ($\beta$) and write $\overset{A}{=}$, when necessary, in the sense $\alpha$, as suggestive of *absolute* error.

Again, to mark the degree of approximation we may write $\overset{1}{=}$, $\overset{2}{=}$, . . . and $\overset{A1}{=}$, $\overset{A2}{=}$, . . . Thus, $U \overset{An}{=}_h V$ or $U \overset{n}{=}_h V$ would mean that $U - V$ or $(U - V)/\sqrt{UV}$ was of at least the $n$th order of smallness or negligeability, $h$ being of the first order, or that it had the factor $h^n$. The suffix could be omitted, and the index $n$ might be fractional or negative: e. g. $\overset{\frac{1}{2}}{=}$. When $n$ is positive, the copula is replaceable by the less explicit $\doteq$ or $\overset{A}{=}$; and any of these copulas may replace their common limiting form $=$.

When $n$ is negative, $U \overset{An}{=} V$ means of course that $U - V$ is not an infinite of more than the $n$th order. But the definition of $\overset{n}{=}$ connects itself naturally with that of "Relative Error." To define the latter in the usual way, as the ratio of the absolute error of an assumed value to the true value, seems to me open to three objections, viz: It ignores the actual symmetry of relation between true and assumed values, assigning different relative errors to an assumed value and its reciprocal or to estimates $k$-fold too large and $k$-fold too small; it makes needlessly inaccurate the rule that the relative error of a product is the sum of the relative errors of the factors; and when the relative errors are far from zero, it may exhibit untruly even their general order of importance, assigning nearly the same relative error, — 1, whether an estimate be .001 or .000001 of the true value. And these objections are of practical importance when, as frequently happens, we would deal with actual magnitudes the probable errors of whose logarithms may be several units.

To meet the first two objections, we must divide the absolute error not by the true value, but by that symmetric function of the true and the assumed value which gives for quotient the absolute error of the (say, Naperian) logarithm of the assumed value. When both values are real, this divisor lies always between their arithmetical and geometric mean and nearer to the geometric, being one third of the way between them when the relative error vanishes, and becoming $\frac{2}{\pi}$ of their arithmetico-geometric mean when the relative error becomes infinite. The divisor hardly differs by over four per cent. from the geometric mean, unless the relative error defined by it exceed 1.

But the third objection is not thus quite removed; for when the true and assumed values are very disproportionate, their ratio tends toward an infinite of infinitely higher order than the relative error as thus defined. Probably we still can best define "Relative Error" as the absolute error of the Naperian logarithm, while defining $\overset{n}{=}$ not quite analogously, but as above, so that $U \overset{n}{\equiv}_h V$ may mean that $\sqrt{U/V} - \sqrt{V/U}$ is no infinite of more that the order $n$, whence the ratio $U/V$ is no infinite or infinitesimal of higher order than the infinitesimal $h^{2n}$: indeed, the two definitions agree as long as $n$ is positive.

If, for a moment, we speak of these approximate equalities as "$R$-equalities" and "$A$-equalities," we can say that:—

(1°) When the two members of an $\left\{\frac{R}{A}\right\}$-equality are $\left\{\begin{matrix}\text{multiplied or}\\ \text{increased or}\end{matrix}\right.$ $\left.\begin{matrix}\text{diminished}\\ \text{divided}\end{matrix}\right\}$ by any same number, or by the corresponding members of another $\left\{\frac{R}{A}\right\}$-equality with the same or higher index, the suffix, whether written or understood, remaining the same; or when they are $\left\{\begin{matrix}\text{raised to any finite}\\ \text{multiplied by any}\end{matrix}\right.$ $\left.\begin{matrix}\text{power}\\ \text{finite number}\end{matrix}\right\}$ whether positive, negative, or imaginary, the index of the resulting $\left\{\frac{R}{A}\right\}$ equality is not diminished; neither is it, if corresponding members be $\left\{\begin{matrix}\text{added, without diminishing}\\ \text{multiplied, without increasing}\end{matrix}\right\}$ the order of magnitude of either member. Here the words "higher," "lower," "increase," "diminution," are, of course, used in their algebraic sense.

(2°) Both members may be integrated between given limits as to any same variable, without diminution of index, unless the real part or the purely imaginary part of a member of the $R$-equality change sign during the integration, or unless a limit of integration for the $A$-equality be infinite. So, both members may be differentiated as to a variable without diminution of index except perhaps for particular values of the variable.

(3°) The suffix may be changed by raising it to any power if the index be divided by the exponent of that power: e. g. the copulas $\overset{n}{=}_{k}$ and $\overset{\frac{1}{2}n}{=}_{k^2}$ and $\overset{2n}{=}_{\sqrt{k}}$ are equivalent.

For example, we may evaluate as follows the elliptic function $\mathbf{p}u$ for $u = 0$. By definition,

$$u \equiv \tfrac{1}{2}\int_{\mathbf{p}u}^{\infty} \frac{dy}{\sqrt{(y^3 - \frac{1}{4}g_2 y - \frac{1}{4}g_3)}}$$

$$\overset{2}{=}_{1/\mathbf{p}} \tfrac{1}{2}\int_{\mathbf{p}u}^{\infty} \frac{dy}{\sqrt{y^3}} = \mathbf{p}^{-\frac{1}{2}} u$$

$$\therefore\ \mathbf{p}u \overset{2}{=}_{1/\mathbf{p}} u^{-2},\ \overset{4}{=}_{u} u^{-2},\ \overset{A2}{=} u^{-2},$$

$$= u^{-2} + \text{terms in } u^2 \text{ and higher powers.}$$

3. *Double Statements*, wherein all the upper or all the lower alternatives must be read together, are of course in common use; and they might often well replace the double column arrangement for exhibiting geometric or other duality. Sometimes there are three or more alternatives.

But occasionally there is for instance a double duality, wherein, to show what cases belong together, accents or other indices would best be used. I do not as yet see, however, that here there are two distinct uses for these marks, assignable respectively to upper and lower positions, as in Arts. 1 and 2.

A good instance of this would be Delambre's fourfold formula for the spherical triangle, when written thus:

$$\overset{\prime}{\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\}}\frac{A \overset{\prime}{\pm} B}{2}\Big/\overset{\prime}{\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}}\frac{C}{2} = \overset{\prime\prime}{\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}}\frac{a \overset{\prime}{\pm} b}{2}\Big/ \overset{\prime\prime}{\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}}\frac{c}{2};$$

wherein we must use all the upper alternatives marked $'$, or else all the lower alternatives marked $'$, and so with the alternatives marked $''$.

When the choice turns upon the evenness or oddness of some number $n$, it is well to associate always the upper alternative with even values, and the lower with odd; writing the number as a suffix if necessary. Thus the general terms of the series $a_0 - a_1 + a_2 - a_3 + \ldots$ and $b_1 - b_2 + b_3 - \ldots$ would be $\pm_n a_n$ and $\pm_{n-1} b_n$, $= \mp_n b_n$, — or simply $\pm a_n$ and $\mp b_n$. With this convention the more cumbrous $(-1)^p$ is seldom needed: thus $a_0 + (-1)^{n-1} b_n$, and $(-1)^m a_m + (-1)^{n-1} b_n$, are simply $a_0 \mp b_n$ and $\pm_m a_m \mp_n b_n$. The laws of combination

[$\pm_m \pm_n$ is $\pm_{m+n}$, $\pm_m \mp_n$ is $\mp_{m+n}$, etc.] are obvious. A like convention would apply, if needed, to any $k$ alternatives dependent upon the form of some number as to $k$; and also to symbols which, like $\sqrt[k]{\ }$, are essentially ambiguous.

Analogous to double statements are statements involving elements joined by the conjunction (,) or (;). Here, too, indices may be sometimes useful; or we may, instead, make like conjunctions, in like type, to correspond. E. g., to express that the substitution of $(lx + my,\ l'x + m'y)$ for $(x, y)$ transforms $(\alpha, \beta, \gamma)(x, y)^2, \equiv \alpha x^2 + 2\beta xy + \gamma y^2$, into $((\alpha, \beta, \gamma)(l, l')^2;\ (\alpha, \beta, \gamma)(l, l')(m, m');\ (\alpha, \beta, \gamma)(m, m')^2)(x, y)^2$, wherein $(\alpha, \beta, \gamma)(l, l')(m, m') \equiv \alpha . lm + \beta(lm' + l'm) + \gamma . l'm'$, one might write

$$(\alpha;\ \beta;\ \gamma)((l, m;\ l', m')(x, y))^2 = ((\alpha, \beta, \gamma)(l, l';\ m, m')^2)(x, y)^2.$$

Restrictions or freedom in the order of correspondence of elements thus joined can be indicated much as with the double statements first considered.

E. g. $$(x, y, z) = (\alpha, \beta, \gamma) = (a, b, c)' = 1,\ 2,\ 3$$

would seem to denote naturally that $x = \alpha$, $y = \beta$, and $z = \gamma$; and that these are equal respectively to $a$, $b$, and $c$ in some order elsewhere specified or determined, and to 1, 2, and 3 in some order that is not specified.

Sometimes it is not clear whether an ordinary double statement stands for two simultaneous statements or only for two alternative ones, and a like doubt may arise when commas or radicals, etc., are used. How best to remedy this would, perhaps, be worth considering.

The general principles of Arts. 1–3 would of course apply, at need, to the modifying of almost any simple or complex sign of quantity, operation, or relation.

4. *Order of members in a continued statement.* (cf. O. W. J., p. 8.)—In the absence of punctuation-marks, each copula in a continued statement naturally connects the two members just adjacent, and of course no others. This is too often forgotten, even by good writers; but, obviously, it is important in stating any chain of equalities or other assertions. A good example is in the above proof of Pascal's theorem.

Sometimes, however, it is convenient, after one or more transformations of a member, to go back, begin a new series of transformations of the first member, and equate the two final results thus obtained. This is naturally indicated by a comma before the first copula that connects back; e. g. $P =_1 Q =_x R,\ =_{23} S$ should mean that $P = Q$ by reason of equation (1), and $P = S$ by (2) and (3), and $Q - R$ approaches the limit 0, or (preferably) $Q/R$ approaches unity, as $x$ approaches its limit; whence $R = S$; while without the comma, $P$ would $= S$, and $R = S$.

So far all seems clear; but some nice questions remain, to which attention is invited. For instance, when a sign of inference precedes an unpunctuated continued statement, does it necessarily affect the first copula of that statement, and only the first copula? E. g., knowing that $\beta < \gamma$, and also, independently, that $\alpha < \beta + 3$, can we write $\beta < \gamma, \quad \therefore \alpha < \beta + 3 < \gamma + 3$; or must it be $\beta < \gamma; \ \therefore \ \alpha, < \beta + 3, < \gamma + 3$, wherein two commas have the effect of a parenthesis?

5. *Other symbols.*—To the above, which relates to the natural modifications and uses of existing symbols, I add, with some diffidence, a brief mention of two or three further devices in general elementary notation; i. e. not in the notation of particular subjects such as derivatives or determinants. Some of them would admit of more development and theory than can be given here.

(*a.*) We need a logical copula to assert the dependence of one statement upon another without our first asserting the truth of either. This, $\therefore$ and $\because$ cannot do; but it is done by Prof. C. S. Peirce's $-\!<$ and $>\!-$; thus $P -\!< Q$, or $Q >\!- P$, means "If $P$ be true, so is $Q$;" and $(A = B) -\!< (C = D)$, or $(C = D) >\!- (A = B)$, means "If $A = B$ then $C = D$." Such a statement neither implies nor denies its own converse; but we might write $P >\!-\!< Q$, for "If $P$ be true, so is $Q$, and, conversely, if $Q$, then $P$;" i. e. "Statements $P$ and $Q$ are equivalent;" and this, or some other sign of equivalence, would often be wanted. Peirce's sign of equivalence is not $>\!-\!<$, but $=$; which is the best in his studies of Pure Logic, but which, of course, is preoccupied in Mathematics. Hence, it may be as well not to employ the forms $-\!<$, $>\!-$, if, as I think is the case, another form (the square half bracket) is already somewhat used for a quite analogous purpose, and is rather more convenient. When we write

$$C = D, \tag*{[(8)}$$

the marginal reference, [(8), is commonly a reason for the statement $C = D$; and if the formula (8), referred to, be $A = B$, we have thus virtually written $C = D \;[A = B$, in the sense $(C = D) >\!- (A = B)$ above; that is, we have the copulas [ "*Is implied by* or necessary to;" ] "*Implies* or is sufficient to;" ][ "*Is equivalent*, or both necessary and sufficient, to." These copulas [, ], ][, or their equivalents $>\!-$, $-\!<$, $>\!-\!<$, can, at need, be variously modified by indices. Thus we could form copulas for "Does not imply," "Implies but is not implied by," etc., analogous to the familiar "Is not equal to." When a logical copula connects only equations not continued and whose second members are zero, we might, for shortness, write only their first members; as, $A + B \;]\, C, D$ for $A + B = 0 ]\; C = D = 0$; i. e. for $A + B = 0 ]\; C = 0, D = 0$.

(*b.*) As it is often convenient to refer to one or more specified members of an equation or other statement, or of a group of such, I denote them by subscripts, as on p. 109 of Vol. III of this journal; e. g. $(3_4)$ would be the fourth member of the continued statement, or group of statements (3); $(3_{1,4})$ would be equivalent to "$(3_1)$ and $(3_4)$;" and $(3_{14})$ or $(3_{135})$ would be the equality $(3_1) = (3_4)$ or $(3_1) = (3_3) = (3_5)$.

Expressions like $(3_1)$ etc., or statements like $(3_{14})$ etc., may of course be used not merely as references, but as elements of new statements or expressions; e. g. of such as are suggested in Art. 5 (*a*).

(*c*) A few other conventions would seem desirable, so as for instance to fix more clearly the order in which the fundamental signs of operation shall combine (cf. O. W. J., I § 12), and to aid in indicating the motives and connection of the different steps in a discussion: but here, at present, I have no definite suggestions to make.